

EXAMPLES OF NON-*FSZ* p -GROUPS FOR PRIMES GREATER THAN THREE

MARC KEILBERG

ABSTRACT. For any prime $p > 3$ and $j \in \mathbb{N}$ we construct examples of non-*FSZ* _{p^j} groups of order p^{p^j+2j-1} . In the special case of $j = 1$ this yields groups of order p^{p+1} , which is the minimum possible order for a non-*FSZ* p -group.

INTRODUCTION

The study of the representation categories of semisimple Hopf algebras, and many other more general contexts, have brought forth an interesting invariant of monoidal categories known as (higher) Frobenius-Schur indicators [1, 2, 3, 5, 9, 12, 13, 14, 15, 16, 17, 19]. These form generalizations of the classical Frobenius-Schur indicators for a finite group G , which for a character χ of G and any $m \in \mathbb{N}$ are defined by

$$(0.1) \quad \nu_m(\chi) = \frac{1}{|G|} \sum_{g \in G} \chi(g^m).$$

When applied to the Hopf algebras $\mathcal{D}(G)$, the Drinfel'd double of the group G —or, more accurately for our purposes, the group algebra of G over \mathbb{C} —, these indicators, while motivated by Hopf algebra considerations, can be expressed entirely in group theoretical terms. Schauenburg [18] has obtained an intriguing description in terms of the character tables of centralizers, in particular. We can therefore consider these indicators as invariants of the underlying group G itself. Frobenius-Schur indicators are guaranteed to be algebraic integers in a certain cyclotomic field, and the Galois action on the field also acts on the indicators [9, Proposition 3.3]. In full generality these indicators need not even be real numbers [9, Example 7.5][7], but in the case of $\mathcal{D}(G)$ they are guaranteed to be real numbers [8, Remark 2.8]. All of the first examples computed in the case of group doubles [4, 10, 11] yielded indicator values in \mathbb{Z} . Since the higher indicators for G itself are classically known to be integers, this raised the question of whether or not the indicators for $\mathcal{D}(G)$ were always integers for arbitrary G .

Iovanov et al. [8] investigated this question, ultimately finding that there were exactly 32 non-isomorphic groups of order 5^6 with non-integer indicators. They dubbed the property of having all integer indicators the *FSZ* property. This property is itself the combination of the properties *FSZ* _{m} for every m , where m plays much the same role as in equation (0.1). Theorem 1.4 below can be taken as a definition of the *FSZ* _{m} properties, and therefore the *FSZ* property. Iovanov et al. [8] also established that several large families of groups were *FSZ*, including but not limited to the symmetric groups S_n ; $PSL_2(q)$ for a prime power q ; and all regular p -groups. On the other hand, the regular wreath product $\mathbb{Z}_p \wr \mathbb{Z}_p$ is an irregular p -group for all primes p , and this was shown to be *FSZ* [8, Example 4.4], thereby

establishing that the class of *FSZ* p -groups properly includes the class of regular p -groups. It is interesting to ask what can be said about the properties of irregular non-*FSZ* p -groups, or alternatively of irregular *FSZ* p -groups.

It is the goal of this note to exhibit an infinite family of non-*FSZ* p -groups for arbitrary primes $p > 3$. The construction, in particular, establishes that there are always non-*FSZ* p -groups of order p^{p+1} when $p > 3$, which is well-known to be the minimum order possible for an irregular p -group.

1. THE CONSTRUCTION

Fix an odd prime p and an integer $j \in \mathbb{N}$.

Consider the abelian p -group

$$P_{p,j} = \mathbb{Z}_{p^{j+1}} \times \mathbb{Z}_p^{p^j-2},$$

with generators a_1, \dots, a_{p^j-1} where a_1 has order p^{j+1} and the rest have order p . We define an endomorphism $b_{p,j}$ of $P_{p,j}$ by

$$\begin{aligned} a_1 &\mapsto a_1 a_2^{-1} \\ a_k &\mapsto a_k a_{k+1}, \quad 1 < k < p^j - 1 \\ a_{p^j-1} &\mapsto a_{p^j-1} a_1^{-p^j}. \end{aligned}$$

It is convenient to write $b_{p,j}$ as a matrix $B_{p,j}$ which acts on the left in the obvious fashion, and whose first row of entries can be taken modulo p^{j+1} and the remaining entries may be taken modulo p . We have

$$B_{p,j} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & -p^j \\ -1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & & & & & & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 1 \end{pmatrix}.$$

The entries of $B_{p,j}^k$ for $1 \leq k \leq p^j - 2$ are then naturally described by the entries $T_{i,j}$ of Pascal's Triangle. For example

$$B_{p,j}^2 = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & -p^j & -2p^j \\ -2 & 1 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 2 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & & & & & & \vdots \\ 0 & 0 & 0 & \cdots & 2 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 2 & 1 \end{pmatrix}$$

and $B_{p,j}^{p^j-2}$ is given by

$$\begin{pmatrix} 1 & -p^j & \cdots & -T_{p^j-1,p^j-3}p^j & -T_{p^j-1,p^j-2}p^j \\ -T_{p^j-1,2} & 1 & \cdots & 0 & 0 \\ -T_{p^j-1,3} & T_{p^j-1,2} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -T_{p^j-1,p^j-2} & -T_{p^j-1,p^j-3} \cdots & & 1 & 0 \\ -1 & T_{p^j-1,p^j-2} & \cdots & T_{p^j-1,2} & 1 \end{pmatrix}.$$

Indeed, the entries of $B_{p,j}^j$ are determined by Pascal's triangle for arbitrary j , just that for $j > p^j - 2$ we can no longer fit entire rows of the triangle in the rows or columns. Nevertheless the pattern is straightforward. The properties of Pascal's triangle then ensure that $b_{p,j}$ is an automorphism of order p^j .

Definition 1.1. Let p be an odd prime and $j \in \mathbb{N}$. Define

$$S(p, j) = P_{p,j} \rtimes \langle b_{p,j} \rangle.$$

This is a group of order p^{p^j+2j-1} .

$S(p, j)$ has center $\langle a_1^p \rangle \cong \mathbb{Z}_{p^j}$. We identify $P_{p,j}$ and $\langle b_{p,j} \rangle$ as subgroups of $S(p, j)$ in the usual fashion, and for simplicity we denote $b_{p,j}$ by simply b whenever convenient, and similarly for $B_{p,j}$.

Now for $0 \leq k < j$ consider the matrices $Y_{p,j}(p^k) = \sum_{m=0}^{p^j-k-1} B^{mp^k}$. Now we have the identity $BY_{p,j}(1) = Y_{p,j}(1)$, so it follows that we have a block decomposition with a 1×1 entry in the upper left corner given by

$$Y_{p,j} = \begin{pmatrix} c * p & 0 \\ 0 & 0 \end{pmatrix}$$

for some integer c . Indeed, it is easily seen that the $(1, 1)$ entry of $B_{p,j}^{p^j-1} = B * B^{p^j-2}$ is exactly $p^j + 1$, from which it follows that the $(1, 1)$ entry of $Y_{p,j}(1)$ is $2p^j$:

$$(1.1) \quad Y_{p,j}(1) = \begin{pmatrix} 2p^j & 0 \\ 0 & 0 \end{pmatrix}.$$

On the other hand, we have

$$(1.2) \quad p^k Y_{p,j}(p^k) = \begin{pmatrix} p^j & 0 \\ 0 & 0 \end{pmatrix}, \quad 1 \leq k < j.$$

These matrices can be used to describe arbitrary p^j -th powers in $S(p, j)$. Namely, fixing $q \in P_{p,j}$ and $b^k \in \langle b \rangle$, then

$$(1.3) \quad (qb^k)^{p^j} = p^t Y_p(p^t)q, \quad |b^k| = p^{j-t}.$$

In particular, all p^j -th powers in $S(p, j)$ yield the subgroup $\langle a^{p^j} \rangle = Z(S(p, j))$

Remark 1.2. The matrices $p^k Y_{p,j}(p^k)$, in particular $Y_{p,j}(1)$, are higher dimensional analogues of the integer parameter d appearing in [10]. The parameter d controlled the existence of negative indicators in the double of the groups under consideration in [10], in much the same way that $Y_{p,j}$ will dictate the existence of non-integer indicators here. More generally, such objects naturally arise when considering the *FSZ* property for groups of the form $A \rtimes C$ where A is abelian and C is cyclic, such as in [8, Example 4.4].

We recall the following.

Definition 1.3. For any group G , $n \in \mathbb{N}$, and $g, u \in G$, define

$$G_n(u, g) = \{a \in G : a^n = (au^{-1})^n = g\}.$$

Of necessity, $G_n(u, g) \neq \emptyset$ implies $[u, g] = 1$, and indeed for fixed g they are subsets of $C_G(g)$. These sets characterize the *FSZ* _{n} properties, as shown by the following.

Theorem 1.4. [8, Corollary 3.2] *Let $n \in \mathbb{N}$ and G a finite group. Then G is an FSZ_n -group if and only if for all commuting pairs of elements u, g and all integers m coprime to $|G|$ we have*

$$|G_n(u, g)| = |G_n(u, g^m)|.$$

Theorem 1.5. *Let notation be as above and set $G = S(p, j)$ for any odd prime $p > 3$ and $j \in \mathbb{N}$. Then $G_{p^j}(ba_1, a_1^{p^j}) = \emptyset$ and $G_{p^j}(ba_1, a_1^{2p^j}) \neq \emptyset$.*

In particular, $S(p, j)$ is non- FSZ_{p^j} .

Proof. We first note that the assumption $p > 3$ is necessary, since when $p = 3$ we have $a_1^{2p^j} = a_1^{-p^j}$ and we always have a bijection $G_n(u, g) \rightarrow G_n(u, g^{-1})$ for any G , $n \in \mathbb{N}$, and $u, g \in G$.

Fix $u = ba_1$ and $Y = \langle a_2, \dots, a_{p^j-1} \rangle$ for the remainder of the proof. Every element $a \in G$ can be written in the form $a = a_1^{j_1} y b^{-k} \in G$ for some $y \in Y$. By equations (1.1) to (1.3) the value of a^{p^j} does not depend on y , so we may suppress elements of Y for the rest of the proof. It follows that when determining the membership of $G_{p^j}(ba_1, g)$ we will naturally break things down into cases, depending on the orders of b^k and b^{k+1} .

First consider the case that $|b^k| = 1$. Then

$$a^{p^j} = a_1^{p^j} = g,$$

while

$$(au^{-1})^{p^j} = X_{p,j}(1)(a_1^{j_1-1}) = a_1^{(j_1-1)p^j} = g.$$

These equalities are consistent if and only if $j_1 \equiv 2 \pmod{p}$. In particular, we have no contribution from elements of this form when $g = a_1^{p^j}$, but do have contributions from such elements when $g = a_1^{2p^j}$.

Now suppose $|b^k| = |b^{k+1}| = p^j$. Then

$$a^{p^j} = X_{p,j}(1)a_1^{j_1} = a_1^{2j_1p^j}$$

and

$$(au^{-1})^{p^j} = X_{p,j}(1)a_1^{j_1} = a_1^{2(j_1-1)p^j}.$$

These values can never be equal, so we have no contributions from elements of this form to the sets $G_{p^j}(u, g)$ for any choice of g .

Next suppose $k \equiv -1 \pmod{p^j}$. Then

$$a^{p^j} = X_{p,j}(1)a_1^{j_1} = a_1^{2j_1p^j}$$

while

$$(au^{-1})^{p^j} = a_1^{(j_1-1)p^j}.$$

These are equal if and only if $j \equiv -1 \pmod{p}$. So when $g \in \{a_1^{p^j}, a_1^{2p^j}\}$ there are no contributions from elements of this form.

For the next case, suppose $|b^k| = p^{j-t}$ for some $0 < t < j$, which implies $|b^{k+1}| = p^j$. It follows that

$$a^{p^j} = a_1^{j_1p^j}$$

and

$$(au^{-1})^{p^j} = a_1^{2(j_1-1)p^j}.$$

These are equal if and only if $j_1 \equiv -2 \pmod{p}$. So for $g \in \{a_1^{p^j}, a_1^{2p_j}\}$ there are no contributions from elements of this form.

Finally, suppose $|b^{k+1}| = p^{j-t}$ for some $0 < t < j$, which implies $|b^k| = p^j$. Then we have

$$a^{p^j} = a_1^{2p^j j_1}$$

while

$$(au^{-1})^{p^j} = a_1^{p^j(j_1-1)}.$$

These values are equal if and only if $j_1 \equiv -1 \pmod{p}$, so again for $g \in \{a_1^{p^j}, a_1^{2p_j}\}$ there are no contributions from elements of this form.

This completes the proof except for the final claim, which follows immediately from Theorem 1.4. \square

Example 1.6. For $p > 3$ we have that $S(p, 1)$ is a group of order p^{p+1} that is not *FSZ* $_p$, and this is the minimum possible order for any non-*FSZ* p -group. Indeed, $S(5, 1)$ is SmallGroup(5⁶, 632) in GAP [6], which is the smallest id number amongst the 32 non-*FSZ* groups of order 5⁶ found by Iovanov et al. [8].

For $p > 3$ and $j > 1$ it is unknown if $S(p, j)$ has minimal order amongst the non-*FSZ* $_{p^j}$ p -groups.

Example 1.7. Iovanov et al. [8] used GAP [6] to verify that there are no non-*FSZ* 2-groups of order at most 2⁹. The author has verified, with the help of the GAP functions in [8, 18], that there are no non-*FSZ* 3-groups of order at most 3⁷. It remains an open question if non-*FSZ* 2-groups or 3-groups exist, and if they do what their minimum orders are. The constructions here, and several attempts at modifications thereof, run into the usual issues for the primes 2, 3.

REFERENCES

- [1] Mohammad Abu-Hamed and Shlomo Gelaki. Frobenius-Schur indicators for semisimple Lie algebras. *J. Algebra*, 315(1):178–191, 2007. ISSN 0021-8693. doi: 10.1016/j.jalgebra.2007.06.003. URL <http://dx.doi.org/10.1016/j.jalgebra.2007.06.003>.
- [2] Peter Bantay. The Frobenius-Schur indicator in conformal field theory. *Phys. Lett. B*, 394(1-2):87–88, 1997. ISSN 0370-2693. doi: 10.1016/S0370-2693(96)01662-0. URL [http://dx.doi.org/10.1016/S0370-2693\(96\)01662-0](http://dx.doi.org/10.1016/S0370-2693(96)01662-0).
- [3] Peter Bantay. Frobenius-Schur indicators, the Klein-bottle amplitude, and the principle of orbifold covariance. *Phys. Lett. B*, 488(2):207–210, 2000. ISSN 0370-2693. doi: 10.1016/S0370-2693(00)00802-9. URL [http://dx.doi.org/10.1016/S0370-2693\(00\)00802-9](http://dx.doi.org/10.1016/S0370-2693(00)00802-9).
- [4] Rebecca Courter. *Computing Higher Indicators for the Double of a Symmetric Group*. PhD thesis, University of Southern California, 2012. arXiv:1206.6908.
- [5] J. Fuchs, A. Ch. Ganchev, K. Szlachányi, and P. Vecsernyés. S_4 symmetry of $6j$ symbols and Frobenius-Schur indicators in rigid monoidal C^* categories. *J. Math. Phys.*, 40(1):408–426, 1999. ISSN 0022-2488. doi: 10.1063/1.532778. URL <http://dx.doi.org/10.1063/1.532778>.
- [6] GAP. *GAP – Groups, Algorithms, and Programming, Version 4.6.5*. The GAP Group, 2013. www.gap-system.org.
- [7] Christopher Goff, Geoffrey Mason, and Siu-Hung Ng. On the gauge equivalence of twisted quantum doubles of elementary abelian and extra-special 2-groups.

- J. Algebra*, 312(2):849–875, 2007. ISSN 0021-8693. doi: 10.1016/j.jalgebra.2006.10.022. URL <http://dx.doi.org/10.1016/j.jalgebra.2006.10.022>.
- [8] M. Iovanov, G. Mason, and S. Montgomery. *FSZ*-groups and Frobenius-Schur indicators of quantum doubles. *Math. Res. Lett.*, 21(4):1–23, 2014.
 - [9] Yevgenia Kashina, Yorck Sommerhäuser, and Yongchang Zhu. On higher Frobenius-Schur indicators. *Mem. Amer. Math. Soc.*, 181(855):viii+65, 2006. ISSN 0065-9266. doi: 10.1090/memo/0855. URL <http://dx.doi.org/10.1090/memo/0855>.
 - [10] Marc Keilberg. Higher indicators for some groups and their doubles. *J. Algebra Appl.*, 11(2):1250030, 38, 2012. ISSN 0219-4988. doi: 10.1142/S0219498811005543. URL <http://dx.doi.org/10.1142/S0219498811005543>.
 - [11] Marc Keilberg. Higher Indicators for the Doubles of Some Totally Orthogonal Groups. *Comm. Algebra*, 42(7):2969–2998, 2014. ISSN 0092-7872. doi: 10.1080/00927872.2013.775651. URL <http://dx.doi.org/10.1080/00927872.2013.775651>.
 - [12] Geoffrey Mason and Siu-Hung Ng. Central invariants and Frobenius-Schur indicators for semisimple quasi-Hopf algebras. *Adv. Math.*, 190(1):161–195, 2005. ISSN 0001-8708. doi: 10.1016/j.aim.2003.12.004. URL <http://dx.doi.org/10.1016/j.aim.2003.12.004>.
 - [13] Sonia Natale. Frobenius-Schur indicators for a class of fusion categories. *Pacific J. Math.*, 221(2):353–377, 2005. ISSN 0030-8730. doi: 10.2140/pjm.2005.221.353. URL <http://dx.doi.org/10.2140/pjm.2005.221.353>.
 - [14] Siu-Hung Ng and Peter Schauenburg. Frobenius-Schur indicators and exponents of spherical categories. *Adv. Math.*, 211(1):34–71, 2007. ISSN 0001-8708. doi: 10.1016/j.aim.2006.07.017. URL <http://dx.doi.org/10.1016/j.aim.2006.07.017>.
 - [15] Siu-Hung Ng and Peter Schauenburg. Higher Frobenius-Schur indicators for pivotal categories. In *Hopf algebras and generalizations*, volume 441 of *Contemp. Math.*, pages 63–90. Amer. Math. Soc., Providence, RI, 2007. doi: 10.1090/conm/441/08500. URL <http://dx.doi.org/10.1090/conm/441/08500>.
 - [16] Siu-Hung Ng and Peter Schauenburg. Central invariants and higher indicators for semisimple quasi-Hopf algebras. *Trans. Amer. Math. Soc.*, 360(4):1839–1860, 2008. ISSN 0002-9947. doi: 10.1090/S0002-9947-07-04276-6. URL <http://dx.doi.org/10.1090/S0002-9947-07-04276-6>.
 - [17] Siu-Hung Ng and Peter Schauenburg. Congruence subgroups and generalized Frobenius-Schur indicators. *Comm. Math. Phys.*, 300(1):1–46, 2010. ISSN 0010-3616. doi: 10.1007/s00220-010-1096-6. URL <http://dx.doi.org/10.1007/s00220-010-1096-6>.
 - [18] P. Schauenburg. Higher frobenius-schur indicators for drinfeld doubles of finite groups through characters of centralizers. *ArXiv e-prints*, April 2016.
 - [19] Peter Schauenburg. On the Frobenius-Schur indicators for quasi-Hopf algebras. *J. Algebra*, 282(1):129–139, 2004. ISSN 0021-8693. doi: 10.1016/j.jalgebra.2004.08.015. URL <http://dx.doi.org/10.1016/j.jalgebra.2004.08.015>.
- E-mail address:* keilberg@usc.edu